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Scattering amplitudes with massive fermions using BCFW recursion

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Abstract. We study the QCD scattering amplitudes for $\bar{q}q \to gg$ and $\bar{q}q \to ggg$ where q is a massive fermion. Using a particular choice of massive fermion spinor we derive compact expressions for the partial spin amplitudes for the $2 \to 2$ process. We then investigate the corresponding $2 \to 3$ amplitudes using the BCFW recursion technique. For the helicity conserving partial amplitudes we present new expressions, but we were unable to treat the helicity-flip amplitudes recursively, except for the case where all the gluon helicities are the same. We therefore evaluate the remaining partial amplitudes using standard Feynman diagram techniques.

1 Introduction

In the last two years there has been dramatic progress in the calculation of multi-particle scattering amplitudes in quantum field theory. Following Witten's [1] discovery of a connection between QCD amplitudes and twistor string theory, a calculational technique [2] was found which has come to be known as 'the CSW construction'. It amounts to an effective scalar perturbation theory, in which MHV amplitudes are elevated to the status of vertices, connected by scalar propagators. This scheme found wide application [3–8], though it turned out that there was an even more efficient way to calculate scattering amplitudes. Britto, Cachazo, Feng and Witten found a recursion relation [9, 10] which, by shifting momenta, takes advantage of the analytic properites of tree amplitudes. Use of the BCFW recursion relation led easily to very compact expressions. Originally applied to purely gluonic tree amplitudes, the recursion has since been extended to include fermions [11, 12], gravitons [13] and loop amplitudes [14–17]. As well as perhaps giving hints of as yet unknown mathematical structure beyond the standard model, these developments are potentially important for the calculation of standard model backgrounds at colliders such as the LHC. The more accurately the relevant cross sections are known, the higher the discovery potential of the machine will be.

One area of very recent progress is the calculation of amplitudes involving massive fermions. It was shown in [18] how to generalize supersymmetric Ward Identities [19, 20] to include massive particles. In this way, different amplitudes involving fields belonging to the same supersymmetric multiplet are related by a rotation. For instance [21], amplitudes involving quarks and gluons are related by SWIs to amplitudes involving scalars and gluons, and these have been calculated in [22]. The off-shell Berends–Giele [23] recursion has also proved useful [24]. Tree amplitudes with massive fermions are required as input within the unitarity [25, 26] method to calculate 1-loop amplitudes, and to this end [27] provides four- and five-point amplitudes with D-dimensional fermions, calculated using BCFW recursion.

The recursion relations were extended in [28] to include massive fermions, and in [29] four-point amplitudes involving two massive quarks and two gluons were calculated. Five-point amplitudes with massive fermions have so far not been treated using BCFW recursion. The goal of the present work is to explore the utility of BCFW recursion to four- and five-point amplitudes with massive fermions. We find that a treatment of massive fermion spinors introduced some twenty years ago in [30] proves to be very useful. We first outline the details of this massive spinor basis, and show that $2 \to 2$ scattering processes in QCD can be written in a form which is ideally suited for use in BCFW recursion. In Sect. 4 we use the recursion relations to derive new, compact expressions for certain $\bar{q}q \to ggg$ partial amplitudes. Those partial amplitudes which we could not treat recursively are evaluated using Feynman diagrams in Sect. 5. Finally, we present our conclusions.

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2 Spinor products

For many years amplitudes involving massless momenta p_i and p_j have been expressed in terms of spinor products,

$$[ij] = \bar{u}^+(p_i)u^-(p_j) \quad \text{and} \quad \langle ij\rangle = \bar{u}^-(p_i)u^+(p_j)\,. \quad (1)$$

In this way amplitudes find their simplest expression. The spinors in question can be thought of either as 2-component Weyl or 4-component Dirac spinors. Numerical evaluation of such amplitudes involves the use of the standard formulae for the spinor products in terms of the momentum 4-vectors. Following [30], let us first introduce two 4-vectors k_0 and k_1 such that

$$k_0\cdot k_0 = 0\,,\quad k_1\cdot k_1 = -1\,,\quad k_0\cdot k_1 = 0\,. \quad (2)$$

We now define a basic spinor $u_-(k_0)$ via

$$u^-(k_0)\bar{u}^-(k_0) = \frac{1-\gamma^5}{2}\not{k}_0\,, \quad (3)$$

and choose the corresponding positive helicity state to be

$$u^+(k_0) = \not{k}_1 u^-(k_0)\,. \quad (4)$$

Using these definitions it is possible to construct spinors for any null momentum p as follows:

$$u^\lambda(p) = \frac{\not{p}\, u^{-\lambda}(k_0)}{\sqrt{2p\cdot k_0}}\,, \quad (5)$$

with $\lambda = \pm$. Note that this satisfies the massless Dirac equation $\not{p}u(p) = 0$, as required. We can now simply evaluate spinor products. For example,

$$\begin{aligned}[ij] &= \bar{u}^+(p_i)u^-(p_j) && (6)\\ &= \frac{(p_i\cdot k_0)(p_j\cdot k_1) - (p_j\cdot k_0)(p_i\cdot k_1) - \mathrm{i}\epsilon_{\mu\nu\rho\sigma}k_0^\mu p_i^\nu p_j^\rho k_1^\sigma}{\sqrt{(p_i\cdot k_0)(p_j\cdot k_0)}}\,. && (7)\end{aligned}$$

A similar expression is obtained for the angle product $\langle ij\rangle$. The arbitrary k_0 and k_1 can now be chosen so as to yield as simple an expression for the product $[ij]$ and $\langle ij\rangle$ as possible, to facilitate numerical evaluation of the amplitudes. The choice[1]

$$\begin{aligned}k_0 &= (1,0,0,1)\,, && (8)\\ k_1 &= (0,0,1,0) && (9)\end{aligned}$$

is a good one, giving the familiar

$$[ij] = (p_i^y + \mathrm{i}p_i^x)\left[\frac{p_j^0 - p_j^z}{p_i^0 - p_i^z}\right]^{\frac{1}{2}} - (p_j^y + \mathrm{i}p_j^x)\left[\frac{p_i^0 - p_i^z}{p_j^0 - p_j^z}\right]^{\frac{1}{2}}\,. \quad (10)$$

[1] The notation is $k^\mu = (k^0, \mathbf{k})$.

2.1 Massive spinors

To evaluate spinor products involving massive spinors, we need to find a definition analogous to (5). One possibility is that outlined in [30],

$$u^\lambda(p) = \frac{(\not{p} + m)u^{-\lambda}(k_0)}{\sqrt{2p\cdot k_0}}\,, \quad (11)$$

which satisfies the massive Dirac equation, $(\not{p} - m)u^\lambda(p) = 0$. The m in (11) is positive or negative when $u^\lambda(p)$ describes a particle or antiparticle respectively. This definition has the virtue[2] of being smooth in the limit $m \to 0$. We will use (11) to evaluate products involving massive spinors.

It is easily seen that the familiar $[.\,.]$ and $\langle.\,.\rangle$ products take the same form for massive spinors as they do for massless ones. Explicit mass terms drop out due to various trace theorems. However, the product of like-helicity spinors is now non-zero:

$$\begin{aligned}(ij) &= \bar{u}^\pm(p_i)u^\pm(p_j) && (12)\\ &= m_i\left(\frac{p_j\cdot k_0}{p_i\cdot k_0}\right)^{\frac{1}{2}} + \mathrm{i} \leftrightarrow j && (13)\\ &= m_i\left(\frac{p_j^0 - p_j^z}{p_i^0 - p_i^z}\right)^{\frac{1}{2}} + i \leftrightarrow j\,, && (14)\end{aligned}$$

where in the last line we have used k_0 as given in (8). Note that the like-helicity product is the same whatever the helicity of the spinors involved, and that we use a round bracket as a shorthand notation for it.

We have been using the word ‘helicity’ to refer to the spin projection of massive fermions, but in fact this is only justified if the projection is onto the direction of the momentum vector, and it is not obvious that this is the case. There exists a unique polarization vector, though it depends on the arbitrary k_0,

$$\sigma^\mu = \frac{1}{m}\left(p^\mu - \frac{m^2}{p\cdot k_0}k_0^\mu\right)\,. \quad (15)$$

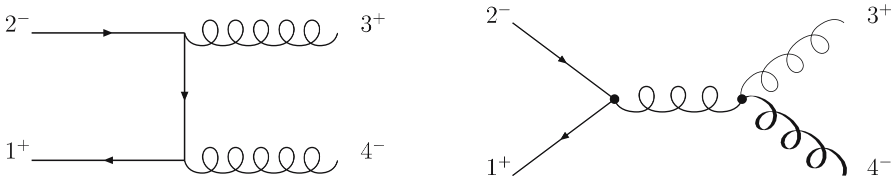
The spinors (11) satisfy

$$\left(1 - \lambda\gamma^5\not{\sigma}\right)u^\lambda = 0\,. \quad (16)$$

We see that besides the momentum p there is an additional contribution to the polarization vector proportional to k_0. Suppose we have an antifermion i and fermion j in the initial state and they approach along the z axis, in the positive and negative directions respectively. If we choose k_0 to be a unit vector in the z direction, i.e.

$$k_0 = \left(1,0,0,1\right)\,, \quad (17)$$

[2] Care is needed when pk_0 also vanishes in this limit, as we will discuss later.



(a) Diagram A (b) Diagram B

Fig. 1. Diagrams contributing to the color-ordered partial amplitude for the process $\bar{q}^+(p_1)q^-(p_2) \to g^+(p_3)g^-(p_4)$

then for the momenta[3]

$$p_i = \big(E, 0, 0, \beta E\big)\,, \tag{18}$$
$$p_j = \big(E, 0, 0, -\beta E\big) \tag{19}$$

we have the following polarization vectors:

$$\sigma_i^\mu = \frac{1}{m_i}\big(-E\beta, 0, 0, -E\big)\,, \tag{20}$$
$$\sigma_j^\mu = \frac{1}{m_j}\big(E\beta, 0, 0, -E\big)\,. \tag{21}$$

If we recall that m_i is negative because i is an antiparticle, then we see that each polarization vector points in the same direction as the corresponding momentum, so that the spinors $u^\lambda(p)$ are indeed helicity eigenstates for this choice of k_0. However, choosing k_0 to be parallel to one of the particle's momenta results, in the massless limit, in the denominators of products such as that in (14) vanishing. By being careful to take the limit algebraically this does not present a problem[4]. But it should be noted that in such cases products like (ij) do not necessarily vanish in the massless limit. We can sidestep this issue by choosing a different k_0, though we could not then talk of the helicity of the fermion.

2.2 Example: $\bar{q}q \to gg$

To demonstrate the use of the massive spinor products described in the previous section we calculate the helicity amplitudes $M^{\lambda_1\lambda_2\lambda_3\lambda_4}$ for the simple QCD process $\bar{q}^{\lambda_1}(p_1)\; q^{\lambda_2}(p_2) \to g^{\lambda_3}(p_3)\; g^{\lambda_4}(p_4)$. The $\lambda_1, \lambda_2 = \pm$ labels on the quarks refer to their spin polarizations in the sense already indicated. If we choose k_0 appropriately then they can be thought of as helicity labels. We will evaluate the partial (color) amplitudes for the above scattering process, i.e. we consider contributions only from those diagrams with a particular ordering of the external gluons. The full color-summed amplitudes can then be recovered by inserting appropriate color factors, as described in Appendix B.

We first consider the M^{+-+-} partial amplitude, for which there are two Feynman diagrams, shown in Fig. 1.

[3] $\beta = \big(1 - \frac{m^2}{E^2}\big)^{1/2}$

[4] If we take $k_0^\mu = (1, 0, \sin\theta, \cos\theta)$, then for the momenta defined in (18) and (19), with $m_j = -m_i = m$, we have $(ij) = -2m\beta\cos\theta(1-\beta^2\cos^2\theta)^{-1/2}$. Thus $(ij) \sim O(m)$ as $m \to 0$ except if $\theta = 0^\circ$ when $(ij) \sim O(E)$.

We will express them in terms of massive spinor products. For the slashed gluon polarization vectors we use

$$\not{\epsilon}^+(p,k) = \sqrt{2}\,\frac{u_+(k)\bar{u}_+(p) + u_-(p)\bar{u}_-(k)}{\langle kp\rangle}\,, \tag{22}$$
$$\not{\epsilon}^-(p,k) = \sqrt{2}\,\frac{u_+(p)\bar{u}_+(k) + u_-(k)\bar{u}_-(p)}{[pk]}\,, \tag{23}$$

where k is a (null) reference vector which may be chosen separately for each gluon. Different choices of the reference vector amount to working in different gauges. The choice $k_3 = p_4$ and $k_4 = p_3$ is particularly convenient in this context, as Diagram B vanishes in this gauge. We have for the other diagram

$$\bar{u}^+(p_1)\,\frac{\not{\epsilon}^-(p_4)}{\sqrt{2}}\,\frac{\not{p}_2 - \not{p}_3 + m}{(p_2-p_3)^2 - m^2}\,\frac{\not{\epsilon}^+(p_3)}{\sqrt{2}}\,u^-(p_2)\,, \tag{24}$$

which simplifies easily to

$$\begin{aligned} &\bar{u}^+(p_3)\not{p}_2 u^+(p_4) \\ &\times \frac{\bar{u}^+(p_1)\big[u^-(p_3)\bar{u}^-(p_4) + u^+(p_4)\bar{u}^+(p_3)\big]u^-(p_2)}{4\,p_3\cdot p_4\;p_4\cdot p_1}\,, \end{aligned} \tag{25}$$

so that

$$M^{+-+-} = [3|2|4\rangle\,\frac{[13](42) + (14)[32]}{4\,p_3\cdot p_4\;p_4\cdot p_1}\,. \tag{26}$$

As promised, we are left with an expression for the amplitude in terms of vector products and massive spinor products.

We next consider the other $M^{\lambda_1\lambda_2+-}$ amplitudes. It is interesting to note that these are directly obtained from the M^{+-+-} amplitude simply by changing the type of certain brackets. Thus

$$M^{+++-} = [3|2|4\rangle\,\frac{[13]\langle 42\rangle + (14)(32)}{4\,p_3\cdot p_4\;p_4\cdot p_1}\,, \tag{27}$$
$$M^{--+-} = [3|2|4\rangle\,\frac{(13)(42) + \langle 14\rangle[32]}{4\,p_3\cdot p_4\;p_4\cdot p_1}\,, \tag{28}$$
$$M^{-++-} = [3|2|4\rangle\,\frac{(13)\langle 42\rangle + \langle 14\rangle(32)}{4\,p_3\cdot p_4\;p_4\cdot p_1}\,. \tag{29}$$

Those amplitudes where the gluons have helicities $(-\,+)$ can be obtained directly from the ones above by complex conjugation.

Let us now examine the case where the gluons have the same helicity. By direct calculation we find

$$M^{--++} = m[43]\,\frac{\langle 13\rangle(42) - \langle 14\rangle(32)}{\langle 34\rangle^2\;2\,p_4\cdot p_1}\,, \tag{30}$$

from which we deduce

$$M^{++++} = m[43]\frac{(13)\langle 42\rangle - (14)\langle 32\rangle}{\langle 34\rangle^2\, 2\, p_4\cdot p_1}\,, \tag{31}$$

$$M^{+-++} = 0\,, \tag{32}$$

$$M^{-+++} = m[43]\frac{\langle 13\rangle\langle 42\rangle - \langle 14\rangle\langle 32\rangle}{\langle 34\rangle^2\, 2\, p_4\cdot p_1}\,, \tag{33}$$

$$= \frac{m[34]\langle 12\rangle}{\langle 34\rangle\, 2\, p_4\cdot p_1}\,, \tag{34}$$

where in the last line we have used the Schouten identity. The amplitudes with two negative helicity gluons are obtained via complex conjugation. There are several interesting things to note about these results. First, the amplitude M^{+-++} vanishes (for any choice of k_0) because of the identity[5] $(13)(42) - (14)(32) = 0$. Second, when k_0 is parallel to the line of approach of the fermions (i.e. when we work in the helicity basis) then the product $\langle 12\rangle$, and hence M^{-+++}, vanishes.

We have verified that when squared and summed over spins and colors, the set of $2\to 2$ scattering amplitudes given above matches the well-known result (see for example [31]) calculated using Feynman diagrams and 'trace technology', namely

$$\sum_{\text{colors}}\sum_{\text{spins}} |M|^2 = 256\left(\frac{1}{6\tau_1\tau_2} - \frac{3}{8}\right) \times\left(\tau_1^2 + \tau_2^2 + \rho - \frac{\rho^2}{4\tau_1\tau_2}\right)\,, \tag{35}$$

where

$$\tau_1 = \frac{2p_1\cdot p_3}{s}\,,\quad \tau_2 = \frac{2p_1\cdot p_4}{s}\,,$$
$$\rho = \frac{4m^2}{s}\,,\quad s = (p_1+p_2)^2\,. \tag{36}$$

Finally, the $m\to 0$ behavior of the spin amplitudes can easily be read off from the expressions given above. For example, if E denotes the typical scale of the $2\to 2$ scattering[6], then in the $m/E\to 0$ limit we have

$$\begin{aligned}
&M^{++\pm\mp},\ M^{--\pm\mp} \sim O(1)\,,\\
&M^{+-\pm\mp},\ M^{-+\mp\pm} \sim O(m/E)\,,\\
&M^{++\pm\pm},\ M^{--\mp\mp} \sim O(m^2/E^2)\,,\\
&M^{+---},\ M^{-+++} \sim O(m/E)\,,\\
&M^{+-++},\ M^{-+--} = 0\,.
\end{aligned} \tag{37}$$

Note that in deriving these results we have assumed that k_0 is *not* directed along any of the particle momenta, so that all (ij) spinor products are $O(m)$ in the $m\to 0$ limit, and $\langle ij\rangle$, $[ij]$ products are $O(E)$. If on the other hand we choose the (fermion) helicity basis by taking k_0 in the direction of (say) p_1, then (37) becomes

$$\begin{aligned}
&M^{+-\pm\mp},\ M^{-+\mp\pm} \sim O(1)\,,\\
&M^{+-\pm\pm},\ M^{-+\mp\mp} = 0\,,\\
&M^{++\pm\mp},\ M^{--\mp\pm} \sim O(m/E)\,,\\
&M^{++++},\ M^{----} \sim O(m/E)\,,\\
&M^{--++},\ M^{++--} \sim O(m^3/E^3)\,.
\end{aligned} \tag{38}$$

3 BCFW recursion

In [9] Britto, Cachazo and Feng introduced new recursion relations for amplitudes involving gluons. The recursion involved *on-shell* amplitudes with momenta shifted by a complex amount. Later [10], the same authors with Witten gave an impressively simple and general proof of the recursion relations. They have since been successfully applied to amplitudes involving fermions [11, 12] and gravitons [13]. Risager [32] has demonstrated how they are related to the earlier 'MHV rules', providing a proof of the latter using an extended form of the BCFW recursion with different shifts to those described below. There has also been much progress at 1-loop level [14–17], which has dovetailed nicely with the earlier unitarity work [25, 26].

We begin by choosing two massless[7] particles i and j whose slashed[8] momenta we shift as follows:

$$\not{p}_i \to \hat{\not{p}}_i = \not{p}_i + z\not{\eta}\,,$$
$$\not{p}_j \to \hat{\not{p}}_j = \not{p}_j - z\not{\eta}\,, \tag{39}$$

where

$$\not{\eta} = u^+(p_j)\bar{u}^+(p_i) + u^-(p_i)\bar{u}^-(p_j) \tag{40}$$

is such that both p_i and p_j remain on-shell. Using the familiar spin-sum condition,

$$\sum_\lambda u^\lambda(p)\,\bar{u}^\lambda(p) = \not{p} \tag{41}$$

we can re-express the shift (39) as a shift of spinors:

$$u^+(p_i) \to u^+(\widehat{p}_i) = u^+(p_i) + z\,u^+(p_j)\,, \tag{42}$$
$$\bar{u}^-(p_i) \to \bar{u}^-(\widehat{p}_i) = \bar{u}^-(p_i) + z\,\bar{u}^-(p_j)\,, \tag{43}$$
$$\bar{u}^+(p_j) \to \bar{u}^+(\widehat{p}_j) = \bar{u}^+(p_j) - z\,u^+(p_i)\,, \tag{44}$$
$$u^-(p_j) \to u^-(\widehat{p}_j) = u^-(p_j) - z\,u^-(p_i)\,. \tag{45}$$

In the Weyl spinor notation we are shifting λ_i and $\tilde{\lambda}_j$. For massless particles, Dirac 4-spinors are effectively two copies of a Weyl 2-spinor, hence the four shifts of (42)–(45). Notice that there is no symmetry between i and j – they are treated differently.

[5] See Appendix A for a list of identities and notation.

[6] We explicitly exclude zero angle scattering.

[7] It should be noted that by only hatting massless external legs we are restricting ourselves to amplitudes with at least two massless particles.

[8] $\not{p} = \gamma^\mu p_\mu$

The amplitude is now a complex function of the parameter z. What the authors of [10] showed was that we can use the analytic properties of the amplitude as a function of z to glean information about the physical case $z = 0$. In particular, we get a recursion relation which can be stated as

$$A_n = \sum_{\text{partitions}} \sum_{\text{s}} A_L(\widehat{p}_i, \widehat{P}^{-s}) \, \frac{1}{P^2 - m_P^2} \, A_R(-\widehat{P}^s, \widehat{p}_j) \,, \tag{46}$$

where the hatted quantities are the shifted momenta. In fact, this is only valid if the helicities of the marked particles are chosen appropriately. The crucial property which must be retained if (46) is to hold is that the shifted amplitude must vanish in the limit $z \to \infty$. There are rules [10, 11, 28, 29] detailing which marking prescriptions are permitted in different cases. For our purposes, we will be on safe ground if the shifted gluons have helicites $(h^i, h^j) = (+,-)$ or $(\pm,\pm)$.

This method of calculation is particularly efficient because much of the computational complexity encountered in a Feynman diagram calculation is avoided since the lower point amplitudes A_L and A_R can be maximally simplified *before* being inserted in (46).

The sum is over all partitions of the particles into a ‘left’ group and a ‘right’ group, subject to the requirement that particles i and j are on opposite sides of the divide. The sum over s is a sum over the spins of the internal particle. Each diagram is associated with a particular value for the complex parameter z, which can be found via the condition that the internal momentum $\widehat{P}$ is on-shell. Note that $\widehat{P}$ is always a function of z because of the restriction that the marked particles i and j appear on opposite sides of the divide.

One useful point to note in practice is that three-point gluon vertices vanish for certain marking choices. In particular, for the j side of the diagram a gluon vertex with helicites $(++-)$ vanishes, as does the combination $(--+)$ on the i side. This was pointed out in [9].

We will be concerned in this work with the process $\bar{q}q \to ggg$, and so we will encounter recursive diagrams connected by an internal fermion, the propagator of which is, in this formalism, the same as that of a scalar. Following [29], we ‘strip’ fermions from the lower point amplitudes which feed the recursion and write

$$A_n = \sum_{\text{partitions}} \sum_{\text{s}} A_L(\widehat{p}_i, \widehat{P}^*) \, \frac{u_s(\widehat{P})\bar{u}_s(\widehat{P})}{P^2 - m_P^2} \, A_R(-\widehat{P}^*, \widehat{p}_j) \,, \tag{47}$$

$$= \sum_{\text{partitions}} A_L(\widehat{p}_i, \widehat{P}^*) \, \frac{\widehat{\not{P}} + m_P}{P^2 - m_P^2} \, A_R(-\widehat{P}^*, \widehat{p}_j) \,. \tag{48}$$

where P^* shows that the amplitude has been stripped of this external spinor wave-function. By way of example, let us reconsider the process $\bar{q}_1^+ q_2^- \to g_3^+ g_4^-$. We mark the gluons such that $i = 3$ and $j = 4$. Then there is one recursive diagram,

$$\bar{u}^+(p_1) \frac{\not{\epsilon}^-(\widehat{p}_4)}{\sqrt{2}} \, \frac{\not{p}_2 - \widehat{\not{p}}_3 + m}{(p_2 - p_3)^2 - m^2} \, \frac{\not{\epsilon}^+(\widehat{p}_3)}{\sqrt{2}} u^-(p_2) \,. \tag{49}$$

With the shifts we have chosen, the hats on the polarization vectors can be removed. The shifted part of the internal propagator is killed by either of the polarization vectors. So in fact all the hats can be removed in (49), which is then identical to the Feynman diagram expression (24).

4 $\bar{q}q \to 3g$ from BCFW recursion

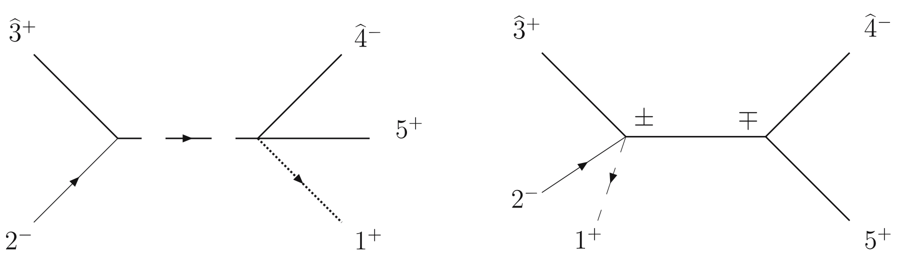
The four-point amplitudes we derived in Sect. 2 are in such a form that it is trivial to strip a fermion off in the manner described above. This means that they are particularly convenient for use in BCFW recursion. Consider the process $\bar{q}_1^+ q_2^- \to g_3^+ g_4^- g_5^+$, for which there are three recursive diagrams, shown in Fig. 2. We choose the marking prescription $i = 3$, $j = 4$.

The two diagrams with internal gluons both vanish, due to the vanishing of M^{+-++} and the vanishing of the $(++-)$ gluon vertex with the shifts we have chosen. For the remaining diagram we use

$$M^{+--+} = -[4|1|3\rangle \frac{(13)[42] + [14](32)}{4\, p_3 \cdot p_4 \; p_4 \cdot p_1} \,, \tag{50}$$

and we strip the fermion $u^-(p_2)$, leaving

$$M^{+\bullet-+} = -[4|1|3\rangle \; \bar{u}^+(p_1) \; \frac{u^+(p_3)\bar{u}^+(p_4) + u^-(p_4)\bar{u}^-(p_3)}{4\, p_3 \cdot p_4 \; p_4 \cdot p_1} \,. \tag{51}$$



(a) Diagram A (b) Diagram B

Fig. 2. Recursive diagrams contributing to $\bar{q}^+(p_1)q^-(p_2) \to g^+(p_3)g^-(p_4)g^+(p_5)$

After the appropriate relabelling this can be used in Diagram A, which can then be written

$$A = -[5|1|\widehat{4}\rangle\ \bar{u}^+(p_1)\ \frac{u^+(\widehat{p_4})\bar{u}^+(p_5) + u^-(p_5)\bar{u}^-(\widehat{p_4})}{4\, p_5 \cdot \widehat{p}_4\, p_5 \cdot p_1} \times \frac{(\not{p}_2 - \widehat{\not{p}}_3 + m)}{(p_2 - p_3)^2 - m^2}\ \frac{\not{\epsilon}^+(\widehat{p_3})}{\sqrt{2}} u^-(p_2)\,. \tag{52}$$

Due to our choice of marking, all the hats in the numerator can be removed. The shifted part of the propagator is killed by the gluon polarization vector. We are left with

$$M^{+-+-+} = [5|1|4\rangle \times \left[\frac{m(14)(42)[53] + [15](42)[3|2|4\rangle - (14)[32][5|1|4\rangle}{8\, p_5 \cdot p_1\, p_2 \cdot p_3\, \widehat{p}_4 \cdot p_5\, \langle 43\rangle}\right]. \tag{53}$$

We can work out z from the requirement that $\widehat{P}^2 = (p_2 - \widehat{p}_3)^2 = m^2$, and we find

$$z = \frac{-2\, p_2 \cdot p_3}{[3|2|4\rangle}\,. \tag{54}$$

The product $\widehat{p}_4 p_5$ is then

$$\widehat{p}_4 \cdot p_5 = (p_4 - z\,\eta) \cdot p_5 \tag{55}$$

$$= p_4 \cdot p_5 + \frac{p_2 \cdot p_3}{[3|2|4\rangle}[3|5|4\rangle\,. \tag{56}$$

The result (53) is much more compact than the expression obtained from a Feynman diagram calculation, with which it agrees. See Sect. 5 for the Feynman results for this process in terms of massive spinor products. We have checked that the expression (53) behaves as expected in the soft gluon limit. That is, as a particular gluon becomes soft, the amplitude factorizes into a $2 \to 2$ amplitude multiplied by a universal 'eikonal factor'.

4.1 Results for helicity conserving amplitudes

Here we give all the helicity conserving QCD amplitudes for $\bar{q}q \to ggg$. By helicity conserving we mean those amplitudes where the spin polarizations of the fermions are $+-$, in the sense described in Sect. 2. Whether these labels actually correspond physically to helicity depends on the choice of k_0. We choose to mark adjacent gluons, so that each amplitude has contributing recursive diagrams of the form of Fig. 2a and b; that is, we have a diagram with an internal fermion and a diagram with an internal gluon. The vanishing of the $2 \to 2$ amplitude M^{+-++} simplifies those cases where there is a majority of positive helicity gluons. In particular, the diagrams with an internal gluon vanish. In the remaining cases, we evaluate such diagrams in the same way as in [9], using identities such as

$$[A\widehat{P}] = \frac{[A|P|i\rangle}{\langle \widehat{P} i\rangle}\,, \tag{57}$$

$$\langle \widehat{P} B\rangle = \frac{[j|P|B\rangle}{[j\widehat{P}]}\,, \tag{58}$$

with i and j as in (39). These identities hold only when A, B and the marked particles i and j are massless.

The results presented here are valid for arbitrary spin polarizations. Choosing a polarization basis amounts to choosing the vector k_0, and when this is done the expressions below will simplify. In the helicity basis for example, in which we choose k_0 to be parallel to the line of approach of the fermions, the building block M^{+---} vanishes. This causes the first term in each of the mostly-minus amplitudes below to vanish:

$$M^{+-+-+} = [5|1|4\rangle \times \left[\frac{m(14)(42)[53] + [15](42)[3|2|4\rangle - (14)[32][5|1|4\rangle}{8\, p_5 \cdot p_1\, p_2 \cdot p_3\, \widehat{p}_4 \cdot p_5\, \langle 43\rangle}\right] \tag{59}$$

where $i = 3$, $j = 4$ and

$$\widehat{p}_4 \cdot p_5 = p_4 \cdot p_5 + \frac{p_2 \cdot p_3}{[3|2|4\rangle}[3|5|4\rangle\,,$$

$$M^{+-++-} = [\widehat{4}|1|5\rangle \times \left[\frac{\begin{pmatrix} m[1\widehat{4}][32]\langle 54\rangle + [1\widehat{4}](42)[3|2|5\rangle \\ +2p_5 \cdot p_1(15)[32] + m(15)(42)[43] \end{pmatrix}}{8\, p_2 \cdot p_3\, \widehat{p}_4 \cdot p_5\, p_5 \cdot p_1 \langle 43\rangle}\right], \tag{60}$$

where $i = 3$, $j = 4$ and

$$\widehat{p}_4 \cdot p_5 = p_4 \cdot p_5 + \frac{p_2 \cdot p_3}{[3|2|4\rangle}[3|5|4\rangle\,,$$

$$|\widehat{4}] = |4] - \frac{(-2\, p_2 \cdot p_3)}{[3|2|4\rangle}|3]\,,$$

$$M^{+--++} = [\widehat{4}|2|3\rangle \times \left[\frac{\begin{pmatrix} m[\widehat{4}2][15]\langle 43\rangle - [\widehat{4}2](14)[5|1|3\rangle \\ -2p_2 \cdot p_3[15](32) + m(14)(32)[54] \end{pmatrix}}{8\, p_2 \cdot p_3\, p_3 \cdot \widehat{p}_4\, p_5 \cdot p_1 \langle 54\rangle}\right], \tag{61}$$

where $i = 5$, $j = 4$ and

$$p_3 \cdot \widehat{p}_4 = p_3 \cdot p_4 + \frac{p_1 \cdot p_5}{[5|1|4\rangle}[5|3|4\rangle\,,$$

$$|\widehat{4}] = |4] - \frac{(-2\, p_1 \cdot p_5)}{[5|1|4\rangle}|5]\,,$$

$$M^{+-+--} = \frac{m[21]\langle 45\rangle^3}{\langle 34\rangle\left[\langle 35\rangle 2p_5 \cdot p_1 + \langle 34\rangle[4|2|5\rangle\right](p_1 + p_2)^2} + [3|2|\widehat{4}\rangle \times \left[\frac{\begin{pmatrix} m(\widehat{4}2)(15)[43] - (\widehat{4}2)[14][3|1|5\rangle \\ -2p_2 \cdot p_3(15)[32] + m[14][32]\langle 54\rangle \end{pmatrix}}{8\, p_2 \cdot p_3\, p_3 \cdot \widehat{p}_4\, p_5 \cdot p_1 [54]}\right], \tag{62}$$

where $i=4$, $j=5$, and

$$p_3\cdot\widehat{p}_4 = p_3\cdot p_4 + \frac{p_1\cdot p_5}{[4|1|5\rangle}[4|3|5\rangle\,,$$

$$(\widehat{42}) = (42) + \frac{(2p_1\cdot p_5)}{[4|1|5\rangle}(52)\,,$$

$$M^{+--+-} = \frac{m[21]\langle 53\rangle^4}{\sqrt{2}\,\langle 43\rangle\langle 45\rangle\left[\langle 53\rangle 2\,p_2\cdot p_3+\langle 54\rangle[4|2|3\rangle\right](p_1+p_2)^2} + [4|1|5\rangle \times\left[\frac{m[14][42]\langle 53\rangle+(15)[42][4|2|3\rangle-[14](32)[4|1|5\rangle}{8\,p_5\cdot p_1\,p_2\cdot p_3\,\widehat{p}_4\cdot p_5[34]}\right]\,, \quad (63)$$

where $i=4$, $j=3$, and

$$\widehat{p}_4\cdot p_5 = p_4\cdot p_5 + \frac{p_2\cdot p_3}{[4|2|3\rangle}[4|5|3\rangle\,,$$

$$M^{+---+} = \frac{m[21]\langle 43\rangle^3}{\sqrt{2}\,\langle 45\rangle\left[\langle 53\rangle 2\,p_2\cdot p_3+\langle 54\rangle[4|2|3\rangle\right](p_1+p_2)^2} + [5|1|\widehat{4}\rangle \times\left[\frac{\begin{pmatrix} m(1\widehat{4})(32)[54]+(1\widehat{4})[42][5|2|3\rangle \\ +2p_5\cdot p_1[15](32)+m[15][42]\langle 43\rangle\end{pmatrix}}{8\,p_2\cdot p_3\,\widehat{p}_4\cdot p_5\,p_5\cdot p_1[34]}\right]\,, \quad (64)$$

where $i=4$, $j=3$, and

$$\widehat{p}_4\cdot p_5 = p_4\cdot p_5 + \frac{p_2\cdot p_3}{[4|2|3\rangle}[4|5|3\rangle\,,$$

$$(1\widehat{4}) = (14) + \frac{(2\,p_2\cdot p_3)}{[4|2|3\rangle}(13)\,,\quad |\widehat{4}\rangle = |4\rangle + \frac{2\,p_2\cdot p_3}{[4|2|3\rangle}|3\rangle\,,$$

$$M^{+----} = \frac{m\langle 5\widehat{4}\rangle[4|2|3\rangle[12][45]}{4\,[45]^2\,p_5\cdot p_1\,p_2\cdot p_3\,[34]}\,, \quad (65)$$

where

$$\langle 5\widehat{4}\rangle = \langle 54\rangle + \frac{2p_2\cdot p_3}{[4|2|3\rangle}\langle 53\rangle\,,$$

$$M^{+-+++} = 0\,. \quad (66)$$

The amplitudes with fermion helicities $-+$ can be obtained from those above by complex conjugation. We have checked that in the soft gluon limit these results factorize as expected.

4.2 Results for helicity-flip amplitudes

We now consider the helicity-flip amplitudes. These have fermion spin polarization labels $\pm\pm$. Here we find diagrams with internal gluons, which cannot be treated with the external-spinor stripping procedure. We must therefore evaluate each side of the diagram directly, which means evaluating spinor products involving the internal momentum. Unfortunately we are unable to evaluate such products as $(\widehat{P}k)$ where k is massive. Here P is the momentum internal to the recursive diagram. In the previous section these products did not occur. Note that in the massless case round brackets do not arise, and any products $\langle\widehat{P}k\rangle$ and $[\widehat{P}k]$ can be evaluated as described in [9]. Those amplitudes in which all gluons have the same helicity do not pose a problem, since the internal gluon diagrams vanish anyway:

$$M^{++---} = m\langle\widehat{43}\rangle \times\left[\frac{m(15)(42)[43]+[14](32)[4|1|5\rangle-[14](42)[3|1|5\rangle}{-4\,p_5\cdot p_1\,p_2\cdot p_3[34]^2[54]}\right]\,, \quad (67)$$

where $i=4$, $j=5$ and

$$|\widehat{4}\rangle = |4\rangle + \frac{2\,p_5\cdot p_1}{[4|1|5\rangle}|5\rangle\,,$$

$$M^{+++++} = m[5\widehat{4}] \times\left[\frac{m(14)(32)\langle 54\rangle+(14)\langle 42\rangle[3|2|5\rangle-(15)\langle 42\rangle[3|2|4\rangle}{-4\,p_5\cdot p_1\,p_2\cdot p_3\langle 45\rangle^2\langle 43\rangle}\right]\,, \quad (68)$$

where $i=3$, $j=4$ and

$$|\widehat{4}] = |4] + \frac{2\,p_2\cdot p_3}{[3|2|4\rangle}|3]\,.$$

We have checked that in the soft gluon limit these results factorize as expected. The amplitudes M^{--+++} and M^{-----} are obtained from those above by complex conjugation. For the remaining amplitudes we resort to Feynman diagrams.

5 Feynman results

Here we give results for $\bar{q}q\to ggg$ derived from Feynman rules. Note that in a given amplitude all the helicities can be flipped by complex conjugation. In all cases where there is overlap, the following expressions agree with BCFW-derived results already given:

$$M^{+--+-} = -[4|2|3\rangle \times\left[\frac{m[14][42]\langle 53\rangle+(15)[42][4|2|3\rangle-[14](32)[4|1|5\rangle}{8\,p_5\cdot p_1\,p_2\cdot p_3\,p_3\cdot p_4\,[54]}\right] + \langle 35\rangle\left[\frac{m[14][42]\langle 53\rangle+(15)[42][4|2|3\rangle-[14](32)[4|1|5\rangle}{8\,p_2\cdot p_3\,p_3\cdot p_4\,p_4\cdot p_5}\right] + \frac{\langle 35\rangle^2}{\langle 34\rangle\langle 45\rangle(p_1+p_2)^2} \times\left[\frac{[14](32)+(13)[42]}{[54]}+\frac{[14](52)+(15)[42]}{[34]}\right]\,, \quad (69)$$

$$
\begin{aligned}
&M^{+-++-}\\
&= -[4|1|5\rangle\\
&\quad\times\left[\frac{-m(15)(52)[43]+(15)[32][4|1|5\rangle-[14](52)[3|2|5\rangle}{8\,p_5\cdot p_1\,p_2\cdot p_3\,p_4\cdot p_5\,\langle 53\rangle}\right]\\
&\quad+[43][4|1|5\rangle\left[\frac{[14](52)+(15)[42]}{4\,p_5\cdot p_1\,p_3\cdot p_4\langle 53\rangle[54]}\right]\\
&\quad-[43]\left[\frac{[14](52)[3|1|5\rangle+(15)[32][4|1|5\rangle-m(15)(52)[43]}{4\,p_5\cdot p_1\,p_3\cdot p_4\,p_4\cdot p_5}\right]\\
&\quad-\frac{[43]^2\langle 35\rangle}{2\,p_3\cdot p_4[54](p_1+p_2)^2}\\
&\quad\times\left[\frac{[14](52)+(15)[42]}{\langle 53\rangle}+\frac{[13](52)+(15)[32]}{\langle 54\rangle]}\right], \qquad (70)
\end{aligned}
$$

$$
\begin{aligned}
&M^{+-+--}\\
&= [3|2|4\rangle\\
&\quad\times\left[\frac{m[13][32]\langle 54\rangle-[13](42)[3|1|5\rangle+(15)[32][3|2|4\rangle}{-8\,p_5\cdot p_1\,p_2\cdot p_3\,p_3\cdot p_4\,[53]}\right]\\
&\quad+\langle 45\rangle[3|2|4\rangle\left[\frac{[13](42)+(14)[32]}{4\,p_2\cdot p_3\,p_4\cdot p_5\langle 43\rangle[35]}\right]\\
&\quad+\langle 45\rangle\left[\frac{[13](42)[3|2|5\rangle+(15)[32][3|2|4\rangle+m[13][32]\langle 54\rangle]}{8\,p_2\cdot p_3\,p_3\cdot p_4\,p_4\cdot p_5}\right]\\
&\quad+\frac{\langle 45\rangle^2[53]}{2\,p_4\cdot p_5\langle 43\rangle(p_1+p_2)^2}\\
&\quad\times\left[\frac{[13](52)+(15)[32]}{[43]}+\frac{[13](42)+(14)[32]}{[53]}\right]. \qquad (71)
\end{aligned}
$$

The corresponding helicity-flip amplitudes can be obtained from these simply by altering the types of brackets. For example, suppose we wish to extract M^{---+-} from M^{+--+-} given above. We can achieve this by changing brackets as follows:

$$[1k]\to(1k)\,, \qquad (72)$$
$$(1k)\to\langle 1k\rangle\,, \qquad (73)$$

where k is massless. Sandwich products such as $[4|1|5\rangle$ are not changed. This transformation results in

$$
\begin{aligned}
&M^{---+-}\\
&= -[4|2|3\rangle\\
&\quad\times\left[\frac{m(14)[42]\langle 53\rangle+\langle 15\rangle[42][4|2|3\rangle-(14)(32)[4|1|5\rangle}{8\,p_5\cdot p_1\,p_2\cdot p_3\,p_3\cdot p_4\,[54]}\right]\\
&\quad+\langle 35\rangle\left[\frac{m(14)[42]\langle 53\rangle+\langle 15\rangle[42][4|2|3\rangle-(14)(32)[4|1|5\rangle}{8\,p_2\cdot p_3\,p_3\cdot p_4\,p_4\cdot p_5}\right]\\
&\quad+\frac{\langle 35\rangle^2}{\langle 34\rangle\langle 45\rangle(p_1+p_2)^2}\\
&\quad\times\left[\frac{(14)(32)+\langle 13\rangle[42]}{[54]}+\frac{(14)(52)+\langle 15\rangle[42]}{[34]}\right]. \qquad (74)
\end{aligned}
$$

Other amplitudes can be found by analogous bracket alterations.

6 Summary

We have calculated all the partial spin amplitudes for the $\bar{q}q\to ggg$ scattering process where q is a massive fermion. For most of the partial amplitudes we were able to use the BCFW recursion relations to obtain fairly compact expressions. This was achieved by following the idea of [29] of stripping lower point amplitudes of their external fermion wave-functions before inserting them into the recursion. We used a particular representation of massive spinors, along the lines of the appendix of [30], to define massive spinor products. In this method information regarding the polarization of the fermion spins is contained in the definition of the spinor products, rather than explicitly in the amplitude.

We derived new, compact results for the *helicity conserving* partial amplitudes. Their simplicity can be attributed to the vanishing of certain $2\to 2$ scattering amplitudes, which reduces the number of contributing recursive diagrams. We were unable to treat the *helicity-flip* amplitudes in the same way (except for the case where all the gluon helicities are the same), since we were unable to evaluate the corresponding recursive diagrams with internal gluons, as in such cases it is not possible to follow the external-spinor stripping procedure. For these amplitudes we instead provided expressions derived from Feynman diagrams, also in terms of massive spinor products. We have confirmed that all the results we have presented have the correct factorization properties in the soft gluon limit. Another useful check is that when the partial amplitudes are combined into a spin-summed cross-section, the result is independent of the vector k_0 used to define fermion polarizations.

These results represent an interesting test of the BCFW recursion relations [9, 10], which have not previously been applied to five-point tree amplitudes with massive fermions. The massive spinor products we used are well suited to such calculations, though there are issues to be resolved (see above). Application of these techniques to higher order processes with massive fermions, such as $\bar{q}q\to gggg$, should be possible though would be accompanied by an increase in complexity. This increase is, however, expected to be significantly less than the corresponding increase in complexity using standard Feynman diagram techniques.

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Appendix A: Notation and conventions

We have used products of Dirac spinors:

$$[ij]=\bar{u}^+(i)u^-(j)\,,\quad \langle ij\rangle=\bar{u}^-(i)u^+(j)\,,$$
$$(ij)=\bar{u}^\pm(i)u^\pm(j)\,, \qquad (A.1)$$

with massive p_i,p_j. To evaluate these we use two arbitrary 4-vectors k_0 and k_1, such that

$$k_0^2 = 0\,, \quad k_1^2 = -1, \quad k_0 \cdot k_1 = 0\,. \tag{A.2}$$

Then

$$[ij] = \frac{(p_i \cdot k_0)(p_j \cdot k_1) - (p_j \cdot k_0)(p_i \cdot k_1) - \mathrm{i}\epsilon_{\mu\nu\rho\sigma} k_0^\mu p_i^\nu p_j^\rho k_1^\sigma}{\sqrt{(p_i \cdot k_0)(p_j \cdot k_0)}}\,, \tag{A.3}$$

$$\langle ij\rangle = \frac{(p_j \cdot k_0)(p_i \cdot k_1) - (p_j \cdot k_1)(p_i \cdot k_0) - \mathrm{i}\epsilon_{\mu\nu\rho\sigma} k_0^\mu p_i^\nu p_j^\rho k_1^\sigma}{\sqrt{(p_i \cdot k_0)(p_j \cdot k_0)}}\,, \tag{A.4}$$

$$(ij) = m_i \left(\frac{p_j \cdot k_0}{p_i \cdot k_0}\right)^{\frac{1}{2}} + i \leftrightarrow j\,, \tag{A.5}$$

where m_i is negative if i is an antiparticle. Different choices of k_0 correspond to different choices of the quantization axis of a massive fermion's spin, as described in Sect. 2.

We use the notation

$$\bar{u}^+(i)\, \not{p}\, u^+(j) = [i|p|j\rangle = [ip]\langle pj\rangle + (ip)(pj)\,, \tag{A.6}$$
$$\bar{u}^-(i)\, \not{p}\, u^-(j) = \langle i|p|j] = \langle ip\rangle[pj] + (ip)(pj)\,. \tag{A.7}$$

Whereas for massless vectors k_i,k_j we have the familiar relation $2k_i \cdot k_j = \langle ij\rangle[ji]$, in the massive case this is extended to

$$2p_i \cdot p_j = \langle ij\rangle[ji] + (ij)^2 - 2m_i m_j\,. \tag{A.8}$$

For any massive i,j and massless k, l we have

$$(ik)(jl) = (il)(jk)\,, \tag{A.9}$$
$$(ik)[li] + [ik](li) = m_i[lk]\,, \tag{A.10}$$
$$\bar{u}^\pm(p_k)\, \not{p}_i\, u^\mp(p_l) = 0\,. \tag{A.11}$$

The Schouten identity holds,

$$\langle a\, b\rangle\langle c\, d\rangle + \langle a\, c\rangle\langle d\, b\rangle + \langle a\, d\rangle\langle b\, c\rangle = 0\,. \tag{A.12}$$

For gluon polarization vectors we use

$$\epsilon_\mu^+(p,k) = \frac{\bar{u}^-(k)\, \gamma_\mu\, u^-(p)}{\sqrt{2}\,\langle kp\rangle}\,, \tag{A.13}$$

$$\epsilon_\mu^-(p,k) = \frac{\bar{u}^+(k)\, \gamma_\mu\, u^+(p)}{\sqrt{2}\,[pk]}\,, \tag{A.14}$$

which take the slashed form

$$\not{\epsilon}^+(p,k) = \sqrt{2}\, \frac{u_+(k)\bar{u}_+(p) + u_-(p)\bar{u}_-(k)}{\langle kp\rangle}\,, \tag{A.15}$$

$$\not{\epsilon}^-(p,k) = \sqrt{2}\, \frac{u_+(p)\bar{u}_+(k) + u_-(k)\bar{u}_-(p)}{[pk]}\,. \tag{A.16}$$

We use a shorthand form for the amplitude in which we display only the helicities of the particles involved. So for example,

$$M(\bar{q}_1^+, q_2^-\,;\, 3^+, 4^-, 5^+) \sim M^{(+-+-+)}\,. \tag{A.17}$$

Appendix B: Color decomposition

The calculation of multi-parton scattering amplitudes in perturbative QCD becomes problematic very quickly as the number of partons increases, due to the sheer number of diagrams and the complicated gluon self-interactions. One technique to circumvent this is to split the set of all Feynman diagrams contributing to a particular amplitude into gauge invariant subsets. Then different gauges can be used in the evaluation of each subset. This simplifies the overall calculation considerably. This is known as color decomposition [33–35]. Each subset of Feynman diagrams is called a *partial amplitude*. We use the normalization

$$\mathrm{Tr}(T^A\, T^B) = \delta_{ab}\,. \tag{B.1}$$

where the T^i are matrices of the fundamental representation of $SU(3)$. This convention leads to color-ordered Feynman rules as given in [36]. The color decomposition for processes with a pair of quarks is then

$$A(\bar{q}, q\,;\, g_1, g_2 s g_n) = \sum_\sigma \left(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(n)}}\right)_{ij} \times M(\bar{q}, q\,;\, g_{\sigma(1)}, g_{\sigma(2)} s g_{\sigma(n)})\,. \tag{B.2}$$

Here σ is the set of all distinct cyclic orderings of the gluons. The color information in a given amplitude is contained purely in the group theoretical prefactors, while all the kinematical information is contained in the partial amplitudes $M(\sigma)$. It is useful to note that amplitudes in QED can be obtained from amplitudes in QCD by replacing all the color matrices T^A with the identity matrix. For a review of color decomposition, the reader is directed to [36].

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